Inverse Problems for Maxwell's Equations in a Slab with Partial Boundary Data.

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Introduction

Let $\Omega = \{x \in \mathbb{R}^3 : 0 < x_3 < L\}$ be a slab in $\mathbb{R}^3$, bounded by $\Gamma_1 = \{x \in \mathbb{R}^3 : x_3 = L\}$ and $\Gamma_2 = \{x \in \mathbb{R}^3 : x_3 = 0\}$.

We fix a frequency $\omega > 0$ and consider the time-harmonic Maxwell equations

$$\nabla \wedge E - i\,\omega\,\mu\,H = 0, \qquad \nabla \wedge H + i\,\omega\,\gamma\,E = 0, \tag{1}$$

where

- E and H are the electric and magnetic fields;
- $\gamma = \varepsilon + i\sigma/\omega$;
- ε - electric permittivity, μ - magnetic permeability, σ - conductivity.

Define the partial Cauchy data set for a bounded subset $\Gamma_1' \subset \Gamma_1$,

$$C^S_{\Gamma_1'}(\mu, \varepsilon, \sigma; \omega) = \left\{ \left((\nu \wedge E)|_{\Gamma_1}, (\nu \wedge H)|_{\Gamma_1'} \right) : (E, H) \text{ solves } (1) \right\},$$

Introduction

Inverse problem: recover the parameters μ, ε, and σ from $C^S_{\Gamma'_1}(\mu, \varepsilon, \sigma; \omega)$.

We prove the following uniqueness result.

Theorem (P. 2018)

Let $\Omega \subset \mathbb{R}^3$ be the slab defined above, and let μ_j, ε_j, $\sigma_j \in C^4(\overline{\Omega})$, $j = 1, 2$, such that:

- *outside of a bounded set B, $\mu_j \equiv \mu_o > 0, \varepsilon_j \equiv \varepsilon_o > 0, \sigma_j \equiv 0$; assume that $\Omega_b := \Omega \cap B$ is a $C^{1,1}$ domain.*
- $\mu_1 = \mu_2, \quad \gamma_1 = \gamma_2$ *up to order one on Γ_1.*
- *there are extensions of the parameters to $\mathbb{R}^3$ belonging to $C^4(\mathbb{R}^3)$ that are invariant under reflection across Γ_2.*

Let $\Gamma'_1 \subset \Gamma_1$ be bounded and such that $B \cap \Gamma_1 \subset \Gamma'_1$. Then if for fixed $\omega > 0$ $C^S_{\Gamma'_1}(\mu_1, \varepsilon_1, \sigma_1; \omega) = C^S_{\Gamma'_1}(\mu_2, \varepsilon_2, \sigma_2; \omega)$, then

$$\mu_1 = \mu_2, \varepsilon_1 = \varepsilon_2, \text{ and } \sigma_1 = \sigma_2 \text{ in } \Omega.$$

Background

Some results for Maxwell's equations on a bounded domain:

[OPS] Ola, Päivärinta, Somersalo (1993), [OS] Ola, Somersalo (1996): first global uniqueness proof for smooth parameters

[COS] Caro, Ola, Salo (2009): partial data inverse problem on special bounded domains, reflection argument

Results for different equations on a slab:

[LU] Li, Uhlmann (2010): uniqueness for the conductivity equation with partial data

[KLU] Krupchyk, Lassas, Uhlmann (2012): Uniqueness for a magnetic Schrödinger operator with partial data

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Review of important tools

An auxiliary elliptic system

Following [OS,COS], we alter Maxwell's equations to an elliptic system:

Introduce scalar functions Φ and Ψ and augment the equations so that $X = (\Phi, H, \Psi, E)^T$ satisfies

$$(P(i\nabla) - V_{\mu,\gamma})X = 0,$$

where $P(i\nabla)$ is the differential operator

$$P(i\nabla) = i \begin{pmatrix} 0 & 0 & 0 & \nabla\cdot \\ 0 & 0 & \nabla & -\nabla\wedge \\ 0 & \nabla\cdot & 0 & 0 \\ \nabla & \nabla\wedge & 0 & 0 \end{pmatrix}$$

and $V_{\mu,\gamma}$ is a matrix potential depending on μ, γ, $\nabla\mu$, and $\nabla\gamma$.

We also rescale the variables as $Y = \mathrm{diag}(\mu^{1/2}\mathrm{I}_4, \gamma^{1/2}\mathrm{I}_4)X$. Then Y satisfies

$$(P(i\nabla) - W_{\mu,\gamma})Y = 0,$$

where $W_{\mu,\gamma}$ also depends on μ, γ, $\nabla\mu$, and $\nabla\gamma$.

Relationship to a Schrödinger operator

This augmentation was done so that

$$\big(P(i\nabla) - W_{\mu,\gamma}\big)\big(P(i\nabla) + W^T_{\mu,\gamma}\big) = -\Delta + \tilde{Q}, \tag{2}$$

where $\tilde{Q}$ is a matrix multiplier depending on μ and γ and their first and second derivatives.

Note: if Z solves

$$\big(-\Delta + \tilde{Q}\big)Z = 0 \quad \text{in} \quad \Omega,$$

then $Y(P(i\nabla) + W^T_{\mu,\gamma})Z$ is a solution to

$$\big(P(i\nabla) - W_{\mu,\gamma}\big)Y = 0,$$

and if $Y_1 = Y_3 = 0$, this solution corresponds to a solution of the original Maxwell's equations.

Complex geometrical optics (CGO) solutions

Let $\zeta \in \mathbb{C}^3$ depend on a large parameter τ. Let $Z_0 = Z_0(\zeta)$ be independent of x and bounded with respect to τ. There exists a CGO solution to

$$(-\Delta + \tilde{Q})Z = 0 \text{ in } \mathbb{R}^3$$

of the form

$$Z(x) = e^{i\zeta\cdot x}(Z_0 + Z_{-1}(x) + Z_r(x)),$$

such that for $-1 < \delta < 0$, and $\delta' > 0$ such that $-1 < \delta + \delta' < 0$, and all $0 \leq |\alpha| \leq 2$,

$$\|\nabla^\alpha Z_{-1}\|_{L^2_\delta} = O(\tau^{-1}), \quad \|\nabla^\alpha Z_r\|_{L^2_\delta} = O(\tau^{|\alpha|-(1+\delta')}).$$

The weighted L^2_δ spaces are defined by

$$\|f\|^2_{L^2_\delta} = \int_{\mathbb{R}^3} (1+|x|^2)^\delta |f(x)|^2 dx.$$

Complex geometrical optics (CGO) solutions

Recall that $Y = (P(i\nabla) + W^T_{\mu,\gamma})Z$ then satisfies $(P(i\nabla) - W_{\mu,\gamma})Y = 0$, and we can write

$$Y(x) = e^{i\zeta\cdot x}(Y_1 + Y_0 + Y_r),$$

such that for $0 \leq \alpha \leq 1$ and for all bounded open subsets U of $\mathbb{R}^3$

$$\|Y_1\|_{L^2(U)} = O(\tau), \quad \|\nabla^\alpha Y_0\|_{L^2(U)} = O(1), \quad \|\nabla^\alpha Y_r\|_{L^2(U)} = O(\tau^{|\alpha|-\delta'}).$$

A suitable choice of Z_0 guarantees that Y yields a solution to Maxwell's equations: If for $k = \omega\sqrt{\mu_0\varepsilon_0}$

$$\big((-P(\zeta) + k)Z_0\big)_1 = \big((-P(\zeta) + k)Z_0\big)_3 = 0,$$

then if τ is sufficiently large,

$$Y_1 = Y_3 = 0.$$

Proof of the Theorem

Integral identity

We have the following identity for solutions to Maxwell's equations with vanishing tangential electric field on the unavailable part Γ_2.

Proposition

Let $X^1 = (0, X_2^1, 0, X_4^1)$ *be an admissible solution to*

$$\big(P(i\nabla) - V_{\mu_1,\gamma_1}\big)X^1 = 0 \quad \text{in} \;\; \Omega,$$

with $\nu \wedge X_4^1 = 0$ *on* Γ_2 *and* $\nu \wedge X_4^1$ *compactly supported on* Γ_1*.*
Let $X^2 = (0, X_2^2, 0, X_4^2)$ *be a solution to*

$$\big(P(i\nabla) - V_{\mu_2,\overline{\gamma_2}}\big)X^2 = 0 \quad \text{in } \Omega_b, \quad \nu \wedge X_4^2 = 0 \text{ on } \partial\Omega_b \cap \Gamma_2.$$

Then

$$\int_\Omega (V_{\mu_2,\gamma_2} - V_{\mu_1,\gamma_1})X^1 \cdot \overline{X^2} dx = 0. \qquad (3)$$

Note: admissibility pertains to requiring solutions to satisfy a partial radiation condition in the unbounded directions.

Runge approximation

Problem: CGO solutions grow exponentially at infinity $\Rightarrow$ not suitable for integral formula on Ω.

Introducing the function spaces

$$\mathcal{W}_1(\Omega) = \left\{ \begin{array}{c} u \in H^1(\Omega)^8 \\ u = (0, u_2, 0, u_4) \end{array} \middle| \begin{array}{c} \left(P(i\nabla) - V_{\mu_1,\gamma_1}\right) u = 0 \text{ in } \Omega,\ \nu \wedge u_4 = 0 \text{ on } \Gamma_2, \\ \nu \wedge u_4 \text{ compactly supported on } \Gamma_1, \\ \text{u is admissible} \end{array} \right\},$$

$$\mathcal{W}_1(\Omega_b) = \left\{ \begin{array}{c} u \in H^1(\Omega_b)^8 \\ u = (0, u_2, 0, u_4) \end{array} \middle| \left(P(i\nabla) - V_{\mu_1,\gamma_1}\right) u = 0 \text{ in } \Omega_b, \nu \wedge u_4 = 0 \text{ on } B \cap \Gamma_2 \right\},$$

the integral identity holds for $X^1 \in \mathcal{W}_1(\Omega)$.
The following lemma allows us to consider $X^1 \in \mathcal{W}_1(\Omega_b)$.

Lemma

$\mathcal{W}_1(\Omega)$ is dense in $\mathcal{W}_1(\Omega_b)$ with respect to the $L^2(\Omega_b)$ norm.

Choice of ζ_1 and ζ_2

The vanishing boundary condition is achieved with a reflection argument.

First pick the phase vectors for the CGO solutions:

Given a fixed $\xi \in \mathbb{R}^3$, $\xi = (\xi_1, \xi_2, \xi_3) = (\xi', \xi_3)$ with $|\xi'| > 0$, we define

$$\eta_1 = \frac{1}{|\xi'|}(\xi_2, -\xi_1, 0), \quad \eta_2 = \eta_1 \wedge \frac{1}{|\xi|}\xi = \frac{1}{|\xi'||\xi|}\big(-\xi_1\xi_3, -\xi_2\xi_3, |\xi'|^2\big).$$

With $k = \omega\sqrt{\mu_0\varepsilon_0}$, let

$$\zeta_1 = \frac{1}{2}\xi + i\Big(\tau^2 + \frac{|\xi|^2}{4}\Big)^{1/2}\eta_1 + \big(\tau^2 + k^2\big)\eta_2, \tag{4}$$

$$\zeta_2 = -\frac{1}{2}\xi - i\Big(\tau^2 + \frac{|\xi|^2}{4}\Big)^{1/2}\eta_1 + \big(\tau^2 + k^2\big)\eta_2, \tag{5}$$

where τ is a large parameter. Note that $i\zeta_1 + \overline{i\zeta_2} = i\xi$.

Reflecting solutions across Γ_2

We have CGO solutions for both sets of equations.

- Let Z^1, Y^1 be the CGO solutions for (μ_1, γ_1) with phase ζ_1;
- let Z^2, Y^2 be the CGO solutions for $(\mu_2, \overline{\gamma_2})$ with phase ζ_2;
- set $X^1 = \mathrm{diag}(\mu_1^{-1/2}\mathrm{I}_4, \gamma_1^{-1/2}\mathrm{I}_4)Y^1$ and $X_2 = \mathrm{diag}(\mu_2^{-1/2}\mathrm{I}_4, \overline{\gamma}_2^{-1/2}\mathrm{I}_4)Y^2$.

We reflect X^1 and X^2 across Γ_2. Denote the reflection across Γ_2 by

$$x = (x_1, x_2, x_3) \mapsto \dot{x}(x) := (x_1, x_2, -x_3)$$

and set

$$\dot{\mathrm{I}}_4 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Reflecting solutions across Γ_2

Then $\dot{X}^1(x) = \mathrm{diag}\left(\dot{\mathrm{I}}_4, -\dot{\mathrm{I}}_4\right) X^1(\dot{x}(x))$ and $\dot{X}^2(x) = \mathrm{diag}\left(\dot{\mathrm{I}}_4, -\dot{\mathrm{I}}_4\right) X^2(\dot{x}(x))$ satisfy

$$(P(i\nabla) - V_{\mu_1,\gamma_1})\dot{X}^1 = 0, \quad (P(i\nabla) - V_{\mu_2,\overline{\gamma_2}})\dot{X}^2 = 0 \ \text{ in } \Omega_b,$$

as well as in $\dot{\Omega}_b = \{\dot{x} : x \in \Omega_b\}$, by our assumption on the parameters.

Furthermore, $\nu \wedge \dot{X}^j_4(x) = -\nu \wedge X^j_4(x)$ on Γ_2, so that

$$\mathbb{X}^1 = X^1 + \dot{X}^1, \quad \mathbb{X}^2 = X^2 + \dot{X}^2$$

satisfy

$$(P(i\nabla) - V_{\mu_1,\gamma_1})\mathbb{X}^1 = 0 \quad \text{in } \Omega_b, \quad \nu \wedge \mathbb{X}^1_4 = 0 \text{ on } \Gamma_2 \cap \partial\Omega_b.$$

$$(P(i\nabla) - V_{\mu_2,\overline{\gamma_2}})\mathbb{X}^2 = 0 \quad \text{in } \Omega_b, \quad \nu \wedge \mathbb{X}^2_4 = 0 \text{ on } \Gamma_2 \cap \partial\Omega_b.$$

Uniqueness of the parameters

We plug $\mathbb{X}^j = X^j + \dot{X}^j$ into the integral formula and express X^j by Y^j. Setting

$$\tilde{\mu} = \omega \frac{\mu_2 - \mu_1}{(\mu_1\mu_2)^{1/2}}, \quad \tilde{\gamma} = \omega \frac{\gamma_2 - \gamma_1}{(\gamma_1\gamma_2)^{1/2}},$$

we get

$$0 = \int_{\Omega_b} \mathrm{diag}\,(\tilde{\mu}\mathrm{I}_4, \tilde{\gamma}\mathrm{I}_4) \left(Y^1 \cdot \overline{Y^2} + Y^1 \cdot \overline{\dot{Y}^2} + \dot{Y}^1 \cdot \overline{Y^2} + \dot{Y}^1 \cdot \overline{\dot{Y}^2} \right) dx.$$

- We perform a change of variables $x \mapsto \dot{x}$ in the last two terms;
- then substitute $Y^1 = \left(P(i\nabla) + W_1^T\right) Z^1$;
- finally integrate by parts to obtain

$$\int_O U\, Z^1 \cdot \overline{(Y^2 + \dot{Y}^2)} dx = 0,$$

where U depends on the parameters and their derivatives, and $O = \Omega_b \cup \dot{\Omega}_b \cup \mathrm{int}(\Gamma_2 \cap \partial\Omega_b)$.

Limit $\tau \to \infty$

Recalling the asymptotics for the solutions

$$Z^1 = e^{i\zeta_1\cdot x}(Z_0^1 + Z_{-1}^1 + Z_r^1), \quad Y^2 = e^{i\zeta_2\cdot x}(Y_1^2 + Y_0^2 + Y_r^2),$$

in the limit $\tau \to \infty$ we are left with (recall that $i\zeta_1 + \overline{i\zeta_2} = i\xi$)

$$0 = \lim_{\tau\to\infty}\int_O e^{i\xi\cdot x} UZ_0^1 \cdot \overline{Y_1^2} dx + \lim_{\tau\to\infty}\int_O e^{i\xi\cdot x} UZ_0^1 \cdot \overline{Y_0^2} dx$$
$$+ \lim_{\tau\to\infty}\int_O e^{i\xi\cdot x} UZ_{-1}^1 \cdot \overline{Y_1^2} dx + \lim_{\tau\to\infty}\int_O UZ^1 \cdot \overline{\dot{Y}^2} dx.$$

The last integral tends to zero. With proper choices of Z_0^1 and Z_0^2, the first three terms become

$$\int_O e^{i\xi\cdot x}\left(\frac{1}{2}\nabla\cdot(\beta_2 - \beta_1) + \frac{1}{4}(\beta_2\cdot\beta_2 - \beta_1\cdot\beta_1) + \kappa_1^2 - \kappa_2^2\right) dx = 0,$$

$$\int_O e^{i\xi\cdot x}\left(\frac{1}{2}\nabla\cdot(\alpha_2 - \alpha_1) + \frac{1}{4}(\alpha_2\cdot\alpha_2 - \alpha_1\cdot\alpha_1) + \kappa_1^2 - \kappa_2^2\right) dx = 0,$$

where $\alpha_j = \nabla \log \gamma_j$, $\beta_j = \nabla \log \mu_j$, and $\kappa_j^2 = \omega^2 \mu_j \gamma_j$.

Uniqueness by unique continuation

So we obtain two partial differential equations for the parameters.

Setting $u = (\gamma_1/\gamma_2)^{1/2}$ and $v = (\mu_1/\mu_2)^{1/2}$, we can rewrite these as

$$-\nabla \cdot (\mu_2 \nabla v) + \omega^2 \mu_2^2 \gamma_2 (u^2 v^2 - 1) v = 0,$$
$$-\nabla \cdot (\gamma_2 \nabla u) + \omega^2 \mu_2 \gamma_2^2 (u^2 v^2 - 1) u = 0.$$

We further have boundary conditions for u and v:

$$u = v = 1, \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial O.$$

A unique continuation property for this system that was proved in [COS] shows that $u = v = 1$ in all of O and completes the uniqueness proof.

Measurements on opposite hyperplanes

The case of opposite hyperplanes

Suppose we measure H on a bounded subset $\Gamma_2' \subset \Gamma_2$, i.e. we have

$$C^D_{\Gamma_2'}(\mu, \varepsilon, \sigma; \omega) = \left\{ \left((\nu \wedge E)|_{\Gamma_1}, (\nu \wedge H)|_{\Gamma_2'} \right) : (E, H) \text{ solves } (1) \right\}.$$

Theorem

Let Ω and Ω_b be as above, and let μ_j, ε_j, $\sigma_j \in C^4(\mathbb{R}^3)$, $j = 1, 2$, be constant in $\Omega \backslash \Omega_b$, and such that

$$\mu_1 = \mu_2, \quad \gamma_1 = \gamma_2 \text{ up to order one on } \partial\Omega.$$

Let $\Gamma_2' \subset \Gamma_2$ such that $\partial\Omega_b \cap \Gamma_2 \subset \Gamma_2'$. Then if for a fixed $\omega > 0$ we have $C^D_{\Gamma_2'}(\mu_1, \varepsilon_1, \sigma_1; \omega) = C^D_{\Gamma_2'}(\mu_2, \varepsilon_2, \sigma_2; \omega)$, then

$$\mu_1 = \mu_2, \ \varepsilon_1 = \varepsilon_2, \text{ and } \sigma_1 = \sigma_2 \text{ in } \Omega.$$

The case of opposite hyperplanes

We get an integral identity with solutions vanishing on different planes.

Proposition

Let $X^1 = (0, X_2^1, 0, X_4^1)$ *be an admissible solution to*

$$\big(P(i\nabla) - V_{\mu_1,\gamma_1}\big)X^1 = 0$$

in Ω*, with* $\nu \wedge X_4^1 = 0$ *on* Γ_2 *and* $\nu \wedge X_4^1$ *compactly supported on* Γ_1*.*
Furthermore, let $X^2 = (0, X_2^2, 0, X_4^2)$ *be a solution to*

$$\big(P(i\nabla) - V_{\mu_2,\overline{\gamma_2}}\big)X^2 = 0 \quad \text{in } \Omega_b, \quad \nu \wedge X_4^2 = 0 \text{ on } B \cap \Gamma_1.$$

Then

$$\int_\Omega (V_{\mu_2,\gamma_2} - V_{\mu_1,\gamma_1})X^1 \cdot \overline{X^2} dx = 0.$$

To construct these solutions, we use reflections across both hyperplanes.

Solutions reflected across Γ_1

- We use different ζ_1 and ζ_2 (following [KLU]) to deal with products of differently reflected functions.

- Denote the reflection across Γ_1 by $\ddot{x}(x) := (x_1, x_2, 2L - x_3)$, and define
$$\ddot{X}^2(x) = \mathrm{diag}\left(\dot{\mathrm{I}}_4, -\dot{\mathrm{I}}_4\right) X^2(\ddot{x}(x)).$$

- $\mathbb{X}^2 = X^2 + \ddot{X}^2$ satisfies $(P(i\nabla) - V_{\mu_2,\overline{\gamma_2}})\mathbb{X}^2 = 0$, and $\nu \wedge \mathbb{X}^2_4 = 0$ on Γ_1.

The integral formula then yields

$$0 = \int_{\Omega_b} \mathrm{diag}\left(\tilde{\mu}\mathrm{I}_4, \tilde{\gamma}\mathrm{I}_4\right)\left(Y^1 \cdot \overline{Y^2} + Y^1 \cdot \overline{\ddot{Y}^2} + \dot{Y}^1 \cdot \overline{Y^2} + \dot{Y}^1 \cdot \overline{\ddot{Y}^2}\right) dx.$$

Plugging CGOs into the integral formula

- Terms with reflected solutions tend to zero:
  By our choice of ζ_1 and ζ_2, the resulting exponential functions decay as τ becomes large.

  For example, the exponent in the last term is

$$i\zeta_1 \cdot \dot{x} + \overline{i\zeta_2} \cdot \ddot{x} = i\dot{\xi} \cdot x + i\xi_3 L - 2\sqrt{\tau^2 - k^2}\frac{|\xi'|}{|\xi|}L.$$

  Since Y_p^j grow at most polynomially in τ, the integral vanishes as $\tau \to \infty$.

- The first term is same as in the previous proof $\Rightarrow$ limits as above.

- We obtain the same set of partial differential equations, so the unique continuation property finishes the proof.

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Thank you!