An inverse problem for Maxwell's equations with Lipschitz parameters.

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Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. We fix a frequency $\omega > 0$ and consider the time-harmonic Maxwell equations

$$\nabla \wedge E - i\,\omega\,\mu\,H = 0, \qquad \nabla \wedge H + i\,\omega\,\gamma\,E = 0, \tag{1}$$

where

- E and H are the electric and magnetic fields;
- $\gamma = \varepsilon + i\sigma/\omega$;
- the parameters ε - electric permittivity, μ - magnetic permeability, σ - conductivity, are assumed to be Lipschitz functions.

We define the Cauchy data set at frequency ω as

$$C(\mu,\varepsilon,\sigma;\omega) = \left\{(\nu\wedge E|_{\partial\Omega}, \nu\wedge H|_{\partial\Omega}) : (E,H) \text{ solves } (1) \text{ with parameters } \mu,\varepsilon,\sigma\right\}.$$

Introduction

We consider the inverse problem of recovering the parameters μ, ε, and σ from $C(\mu, \varepsilon, \sigma; \omega)$. Our aim is to prove the following uniqueness result.

Theorem (P. 2018)

Let $\Omega \subset \mathbb{R}^3$ be a non-empty bounded domain such that $\partial\Omega$ is locally described by the graph of a Lipschitz function. Fix $\omega > 0$, and let $\mu_1,\ \varepsilon_1,\ \sigma_1,\ \mu_2,\ \varepsilon_2,\ \sigma_2 \in C^{0,1}(\overline{\Omega})$ be bounded Lipschitz functions such that for a positive constant c_o and $j = 1, 2$,

$$|\mu_j(x) - \mu_j(y)| \le c_o|x - y|, \quad |\varepsilon_j(x) - \varepsilon_j(y)| \le c_o|x - y|, \quad |\sigma_j(x) - \sigma_j(y)| \le c_o|x - y|,$$

$$0 < \mu_o \le \mu_j(x), \quad 0 < \varepsilon_o \le \varepsilon_j(x), \quad 0 \le \sigma_j(x) \quad \forall x \in \overline{\Omega}.$$

Assume further that $\mu_1(x) = \mu_2(x)$, $\varepsilon_1(x) = \varepsilon_2(x)$, and $\sigma_1(x) = \sigma_2(x)$ for all $x \in \partial\Omega$.

Then if $C(\mu_1, \varepsilon_1, \sigma_1; \omega) = C(\mu_2, \varepsilon_2, \sigma_2; \omega)$, it follows that $\mu_1 = \mu_2,\ \varepsilon_1 = \varepsilon_2$, and $\sigma_1 = \sigma_2 \ \text{ in } \ \Omega$.

Background

Previous uniqueness results for the inverse problem for Maxwell's equations:

[OPS] Ola, Päivärinta, Somersalo (1993), [OS] Ola, Somersalo (1996): uniqueness for C^∞ parameters on C^∞ domain

[CZ] Caro, Zhou (2014): Uniqueness for continuously differentiable parameters on a Lipschitz domain

These results employ tools originally developed for the Calderón problem:

[SU] Sylvester, Uhlmann (1987): uniqueness for C^∞ conductivity on C^∞ domain

[HT] Haberman, Tataru (2013): Uniqueness for continuously differentiable conductivity on a Lipschitz domain

[CR] Caro, Rogers (2016): Uniqueness for Lipschitz conductivity on a Lipschitz domain

Overview

An auxiliary elliptic system

Integral formula

a priori estimate

Construction of CGO solutions

- Construction of w_1
- Construction of v_2

Uniqueness of the parameters

An auxiliary elliptic system

An auxiliary elliptic system

Following [OS] approach, augment (1) to obtain an elliptic system: set

$$\Phi = \frac{i}{\omega}\nabla\cdot(\gamma E)\,, \qquad \Psi = \frac{i}{\omega}\nabla\cdot(\mu H)\,.$$

and

$$e = \gamma^{1/2}E, \quad h = \mu^{1/2}H, \quad \phi = \frac{1}{\gamma\mu^{1/2}}\Phi, \quad \psi = \frac{1}{\gamma^{1/2}\mu}\Psi,$$

then $X = (\phi, e, h, \psi)^T$ is a weak solution to the matrix differential equation

$$(P(i\nabla) - k + V)X = 0,$$

where $k = \omega(\mu_o\varepsilon_o)^{1/2}$, $P(i\nabla)$ is the elliptic first order matrix differential operator

$$P(i\nabla) = i\begin{pmatrix} 0 & \nabla\cdot & 0 & 0 \\ \nabla & 0 & \nabla\wedge & 0 \\ 0 & -\nabla\wedge & 0 & \nabla \\ 0 & 0 & \nabla\cdot & 0 \end{pmatrix},$$

and V is a weakly defined potential depending on μ, γ, $\nabla\mu$, and $\nabla\gamma$.

An auxiliary elliptic system

With $\mathcal{P} := P(i\nabla) - k + V$, $\mathcal{P}' := P(i\nabla) + k - V^T$, we have

$$\mathcal{P}\mathcal{P}' = -(\Delta + k^2) + Q \quad \text{and} \quad \mathcal{P}'\mathcal{P} = -(\Delta + k^2) + \tilde{Q},$$

where Q and $\tilde{Q}$ are weakly defined matrix multipliers.

Note that if w is a weak solution to the second order equation

$$\big[-(\Delta + k^2) + Q\big] w = 0 \quad \text{in} \quad \Omega,$$

then $v = \mathcal{P}' w = (P(i\nabla) + k - V^T)w$ is a weak solution to

$$(P(i\nabla) - k + V)v = 0,$$

and if $v_1 = v_4 = 0$, this solution corresponds to a solution of the original Maxwell's equations.

Integral formula

Integral formula

We extend the two sets of parameters to Lipschitz functions on all of $\mathbb{R}^3$ such that $\mu_1 = \mu_2$, $\varepsilon_1 = \varepsilon_2$, and $\sigma_1 = \sigma_2$ outside Ω, and outside a sufficiently large ball $\Omega' \supset \Omega$, $\mu_1 = \mu_2 = \mu_o$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_o$, and $\sigma_1 = \sigma_2 = 0$.

Proposition

Let $w_1 \in H^1_{loc}(\mathbb{R}^3)^8$ be a weak solution to the equation $[-(\Delta + k^2) + Q_1]w_1 = 0$ in the bounded domain Ω', and assume that $v_1 = \mathcal{P}'_1 w_1$ has vanishing first and last components. Furthermore, let $v_2 \in H^1_{loc}(\mathbb{R}^3)^8$ satisfy

$$\left(P(i\nabla) + k - V_2^T\right)v_2 = 0 \quad \text{in } \Omega'. \tag{2}$$

Then, if the Cauchy data sets $C(\mu_1, \varepsilon_1, \sigma_1; \omega)$ and $C(\mu_2, \varepsilon_2, \sigma_2; \omega)$ on $\partial\Omega$ are equal, the following integral identity holds:

$$\left\langle (Q_2 - Q_1)w_1, v_2 \right\rangle = 0. \tag{3}$$

Complex geometrical optics solutions

Our next goal is to construct suitable *complex geometrical optics* (CGO) solutions w_1 and v_2, i.e. solutions of the form

$$w_1(x) = e^{\zeta_1 \cdot x}(A_{\zeta_1} + R_{\zeta_1}(x)), \quad v_2(x) = e^{\zeta_2 \cdot x}(B_{\zeta_2} + S_{\zeta_2}(x)),$$

where $\zeta_1, \zeta_2 \in \mathbb{C}^3$ are complex phase vectors depending on a large parameter s, $A_{\zeta_1}, B_{\zeta_2} \in \mathbb{C}^8$ are constant vectors, and R_{ζ_1} and S_{ζ_2} decay in a suitable sense as s becomes large.

Plugging this ansatz for w_1 into the Schrödinger equation, we obtain the following equation for R_{ζ_1}:

$$(-\Delta - (2\zeta_1 \cdot \nabla)I_8 + Q_1)R_{\zeta_1} = -Q_1 A_{\zeta_1}. \tag{4}$$

a priori estimate

a priori estimate

We will construct solutions w_1 and v_2 that belong to the following weighted spaces, which were first introduced in [HT].

Denote by p_ζ the polynomial

$$p_\zeta(\xi) = |\xi|^2 - 2i\zeta \cdot \xi.$$

For $b \in \mathbb{R}$, we define X_ζ^b to be the closure of the set of functions $w \in \mathcal{S}'(\mathbb{R}^3)^8$ for which

$$\|w\|_{X_\zeta^b} = \left(\sum_{j=1}^{8} \left\| \left(|\zeta| + |p_\zeta| \right)^b \hat{w}^j \right\|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2}$$

is finite, where $\hat{w}$ denotes the Fourier transform of w.

We will use these spaces with $b = \pm\frac{1}{2}$.

a priori estimate

The following estimate is an adaptation of a scalar estimate derived in [CR] for constructing solutions to the conductivity equation.
We set $\alpha = \nabla \log \gamma$ and $\beta = \nabla \log \mu$.

Theorem

Let $\zeta \in \mathbb{C}^3$ such that $\mathrm{Re}\,\zeta \perp \mathrm{Im}\,\zeta$ and $|\mathrm{Re}\,\zeta|^2 = \tau^2 = |\mathrm{Im}\,\zeta|^2 - k^2$. Furthermore, fix a constant $A > \max\{\|\alpha\|_{L^\infty}, \|\beta\|_{L^\infty}, 1\}$. Then there exists an absolute constant C such that for $\tau > CR^3A^4$,

$$\|u\|_{X_\zeta^{1/2}} \lesssim \|(-\Delta + 2\zeta \cdot \nabla + Q)\, u\|_{X_\zeta^{-1/2}}, \tag{5}$$

provided that $u \in \mathcal{S}(\mathbb{R}^3)^8$ with $\mathrm{supp}\, u \subset \{|x| \leq R\}$. The implicit constant depends on A and R.

a priori **estimate**

Since this estimate holds only for compactly supported functions, we consider the following localized spaces:

$$X^b_\zeta(\Omega') = \{u|_{\Omega'} : u \in X^b_\zeta\}, \quad b > 0,$$

with the norm

$$\|u\|_{X^b_\zeta(\Omega')} = \inf\{\|v\|_{X^b_\zeta} : u = v|_{\Omega'}\},$$

as well as

$$X^b_{\zeta,c}(\Omega') = \{u \in X^b_\zeta : \operatorname{supp} u \subset \overline{\Omega'}\}, \quad b \in \mathbb{R},$$

with the norm of X^b_ζ. Note that $X^b_{\zeta,c}(\Omega')$ is a Hilbert space, and we can define $X^{-b}_\zeta(\Omega')$ to be its dual space.

Construction of CGO solutions

Construction of CGO solutions

Recall that we are looking for solutions of the form

$$w_1(x) = e^{\zeta_1 \cdot x}(A_{\zeta_1} + R_{\zeta_1}(x)), \quad v_2(x) = e^{\zeta_2 \cdot x}(B_{\zeta_2} + S_{\zeta_2}(x)).$$

We pick the phase vectors ζ_1 and ζ_2 as follows. Fix $\rho \in \mathbb{R}^3$, and choose unit vectors $\eta_1, \eta_2 \in \mathbb{R}^3$ such that $\{\rho, \eta_1, \eta_2\}$ is an orthogonal basis of $\mathbb{R}^3$. Let $s \in \mathbb{R}$ with $s \geq 1$. Set

$$\zeta_1 = -\sqrt{s^2 + \frac{|\rho|^2}{4}}\,\eta_1 + i\left(\frac{1}{2}\rho - \sqrt{s^2 + k^2}\,\eta_2\right),$$

$$\zeta_2 = \sqrt{s^2 + \frac{|\rho|^2}{4}}\eta_1 + i\left(\frac{1}{2}\rho + \sqrt{s^2 + k^2}\eta_2\right).$$

Note that $\zeta_j \cdot \zeta_j = -k^2$ for $j = 1, 2$, and $\zeta_1 + \zeta_2 = i\rho$.

Construction of R_{ζ_1}

Recall that the ansatz for w_1 yields the following weakly satisfied equation for R_{ζ_1},

$$(-\Delta - (2\zeta_1 \cdot \nabla) I_8 + Q_1) R_{\zeta_1} = -Q_1 A_{\zeta_1},$$

that is, for any $\varphi \in (\mathcal{S}(\mathbb{R}^3))^8$ with $\operatorname{supp} \varphi \subset \Omega'$,

$$\langle R_{\zeta_1}, (-\Delta + (2\zeta_1 \cdot \nabla) I_8 + Q_1)\varphi \rangle = -\langle Q_1 A_{\zeta_1}, \varphi \rangle . \tag{6}$$

Using the *a priori* estimate and the Hahn Banach theorem, we can find a solution $R_{\zeta_1} \in X^{1/2}_{\zeta_1,c}(\Omega')$ that decays in the averaged sense introduced in [HT].

Lemma

For given ζ_1 as above, there exists a solution $R_{\zeta_1} \in X^{1/2}_{\zeta_1,c}(\Omega')$ to (6). Furthermore, R_{ζ_1} satisfies the averaged estimate

$$\frac{1}{\lambda} \int_{S^1} \int_{\lambda}^{2\lambda} \|R_{\zeta_1}\|^2_{X^{1/2}_{\zeta_1}} \, ds \, d\eta_1 = o\big(\mathbf{1}(\lambda)\big), \quad \lambda \to \infty. \tag{7}$$

Sketch of proof

Define the linear subspace

$$\mathcal{L} = \left\{(-\Delta + (2\zeta_1 \cdot \nabla) I_8 + Q_1)\,\varphi : \varphi \in (\mathcal{S}(\mathbb{R}^3))^8,\ \operatorname{supp}\varphi \subset \Omega'\right\} \subset X_{\zeta_1}^{-1/2}(\Omega'),$$

and the linear functional L on $\mathcal{L}$ by $Lv = -\langle Q_1 A_{\zeta_1}, \varphi\rangle$, where φ is such that $v = (-\Delta + (2\zeta_1 \cdot \nabla) I_8 + Q_1)\varphi$.

By the *a priori* estimate we have $\|L\| \leq \|Q_1 A_{\zeta_1}\|_{X_{\zeta_1}^{-1/2}}$, so that the Hahn Banach Theorem allows to extend it to all of $X_{\zeta_1}^{-1/2}(\Omega')$.

By duality there is $R_{\zeta_1} \in X_{\zeta_1,c}^{1/2}(\Omega')$ with $\|R_{\zeta_1}\|_{X_{\zeta_1}^{1/2}} \lesssim \|Q_1 A_{\zeta_1}\|_{X_{\zeta_1}^{-1/2}}$ such that

$$Lv = \langle R_{\zeta_1}, v\rangle \quad \text{for } v \in X_{\zeta_1}^{-1/2}(\Omega')$$

The averaged estimate for R_{ζ_1} now follows from the estimate

$$\frac{1}{\lambda}\int_{S^1}\int_{\lambda}^{2\lambda} \|Q_1 A_{\zeta_1}\|^2_{X_{\zeta_1}^{-1/2}}\, ds\, d\eta_1 = o(\mathbf{1}(\lambda)).$$

which is derived using estimates from [HT]. □

Construction of w_1

We extend $R_{\zeta_1} \in X^{1/2}_{\zeta_1,c}(\Omega')$ by zero into a slightly bigger bounded domain $\Omega'' \supset \overline{\Omega'}$, yielding a compactly supported function in $X^{1/2}_{\zeta_1}(\Omega'')$. Since $Q_1 = 0$ outside Ω', this extension is in fact a solution to (4) in Ω''.

The solution w_1 is suitable for the integral formula if $v = \mathcal{P}_1' w_1$ has vanishing first and last components. This can be guaranteed by a careful choice of the constant vector A_{ζ_1}. Writing out the expression for v_1 yields

$$\begin{aligned} v_1 =& e^{\zeta_1 \cdot x} \Big(\underbrace{i\zeta_1 \cdot A_{\zeta_1,2} + kA_{\zeta_1,1}}_{=const} \\ & + \underbrace{i(\nabla + \zeta_1) \cdot R_{\zeta_1,2} + kR_{\zeta_1,1} - (k-\kappa)(R_{\zeta_1,1} + A_{\zeta_1,1}) - \frac{i}{2}\beta \cdot (R_{\zeta_1,2} + A_{\zeta_1,2})}_{=S} \Big), \end{aligned}$$

and we pick A_{ζ_1} so that the constant part vanishes. The remainder vanishes on the boundary of Ω'' and from this we can conclude that $v_1 = 0$.

Construction of v_2

We next construct $v_2(x) = e^{\zeta_2\cdot x}\left(B_{\zeta_2} + S_{\zeta_2}(x)\right)$ solving $\mathcal{P}_2' v_2 = 0$ in Ω'.

We first look for w_2 solving $\mathcal{P}_2'\mathcal{P}_2 w_2 = [-(\Delta + k^2) + \tilde{Q}_2]w_2 = 0$ and then set $v_2 = \mathcal{P}_2 w_2$.

Proposition

For $|\zeta_2|$ sufficiently large, there is a solution $w_2 \in H^1(\Omega')^8$ of the form $w_2(x) = e^{\zeta_2\cdot x}(A_{\zeta_2} + R_{\zeta_2}(x))$, with $R_{\zeta_2} \in X^{1/2}_{\zeta_2,c}(\Omega')$, to

$$[-(\Delta + k^2) + \tilde{Q}_2]w_2 = 0 \quad \text{in } \Omega'.$$

The function $v_2 = \mathcal{P}_2 w_2$ belongs to $H^1(\Omega'')^8$ and is a weak solution to

$$\left(P(i\nabla) + k - V_2^T\right)v_2 = 0 \quad \text{in } \Omega''.$$

We can extend v_2 to $H^1_{loc}(\mathbb{R}^3)^8$ and write $v_2(x) = e^{\zeta_2\cdot x}(B_{\zeta_2} + S_{\zeta_2}(x))$, with constant B_{ζ_2} and $S_{\zeta_2} \in X^{1/2}_{\zeta_2}$, where S_{ζ_2} satisfies

$$\frac{1}{\lambda}\int_{S^1}\int_{\lambda}^{2\lambda} \|S_{\zeta_2}\|^2_{X^{1/2}_{\zeta_2}} ds\, d\eta_1 = o(1), \quad \lambda \to \infty.$$

Uniqueness of the parameters

Uniqueness of the parameters

Using w_1 and v_2 in the integral formula and noting that $e^{\zeta_1\cdot x}e^{\zeta_2\cdot x} = e^{i\rho\cdot x}$ we get

$$\left\langle (Q_2 - Q_1)e^{i\rho\cdot x}(R_{\zeta_1} + A_{\zeta_1}), S_{\zeta_2} + B_{\zeta_2} \right\rangle = 0.$$

With $A_1 = \lim_{s\to\infty} A_{\zeta_1}$ and $B_2 = \lim_{s\to\infty} B_{\zeta_2}$, we rewrite this to obtain

$$\begin{aligned} -\left\langle (Q_2 - Q_1)e^{i\rho\cdot x}A_1, B_2 \right\rangle &= \left\langle (Q_2 - Q_1)(R_{\zeta_1} + A_{\zeta_1}), e^{i\rho\cdot x}(B_{\zeta_2} - B_2 + S_{\zeta_2}) \right\rangle \\ &\quad + \left\langle (Q_2 - Q_1)B_2, e^{i\rho\cdot x}(R_{\zeta_1} + A_{\zeta_1} - A_1) \right\rangle. \end{aligned}$$

Letting $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\chi = 1$ on Ω', we can estimate this as

$$\begin{aligned} &\left| \left\langle (Q_2 - Q_1)A_1 e^{i\rho\cdot x},\, B_2 \right\rangle \right| \\ &\qquad \lesssim \left(\|(Q_2 - Q_1)A_{\zeta_1}\|_{X_{\zeta_2}^{-1/2}} + \|R_{\zeta_1}\|_{X_{\zeta_1}^{1/2}} \right)\left(\|(B_{\zeta_2} - B_2)\chi\|_{X_{\zeta_2}^{1/2}} + \|S_{\zeta_2}\|_{X_{\zeta_2}^{1/2}} \right) \\ &\qquad\quad + \|(Q_2 - Q_1)B_2\|_{X_{\zeta_1}^{-1/2}}\left(\|R_{\zeta_1}\|_{X_{\zeta_1}^{1/2}} + \|(A_{\zeta_1} - A_1)\chi\|_{X_{\zeta_1}^{1/2}} \right). \end{aligned}$$

Uniqueness of the parameters

Taking the average of $(s, \eta_1) \in [\lambda, 2\lambda] \times S^1$ and letting $\lambda \to \infty$, we obtain

$$\Big(o(\mathbf{1}(\lambda)) + o(\mathbf{1}(\lambda))\Big)\Big(O(\mathbf{1}(\lambda)) + o(\mathbf{1}(\lambda))\Big) + o(\mathbf{1}(\lambda))\Big(o(\mathbf{1}(\lambda)) + O(\mathbf{1}(\lambda))\Big).$$

Thus, the right-hand side tends to 0 on average, and the left-hand side is independent of λ, hence

$$\big\langle (Q_2 - Q_1)e^{i\rho\cdot x}A_1, B_2\big\rangle = 0.$$

Using different suitable choices of A_{ζ_1} and B_{ζ_2}, we arrive at the equations

$$\frac{1}{4}\Big(\frac{\nabla\gamma_2}{\gamma_2} - \frac{\nabla\gamma_1}{\gamma_1}\Big)\cdot\Big(\frac{\nabla\gamma_2}{\gamma_2} + \frac{\nabla\gamma_1}{\gamma_1}\Big) - \omega^2(\gamma_2\mu_2 - \gamma_1\mu_1) + \frac{1}{2}\nabla\cdot\Big(\frac{\nabla\gamma_2}{\gamma_2} - \frac{\nabla\gamma_1}{\gamma_1}\Big) = 0,$$
$$\frac{1}{4}\Big(\frac{\nabla\mu_2}{\mu_2} - \frac{\nabla\mu_1}{\mu_1}\Big)\cdot\Big(\frac{\nabla\mu_2}{\mu_2} + \frac{\nabla\mu_1}{\mu_1}\Big) - \omega^2(\gamma_2\mu_2 - \gamma_1\mu_1) + \frac{1}{2}\nabla\cdot\Big(\frac{\nabla\mu_2}{\mu_2} - \frac{\nabla\mu_1}{\mu_1}\Big) = 0.$$

Uniqueness of the parameters

These equations can be rewritten, with $f := \gamma_2^{1/2} - \gamma_1^{1/2}$ and $g := \mu_2^{1/2} - \mu_1^{1/2}$, as

$$-\Delta f + Wf + af + bg = 0,$$
$$-\Delta g + \tilde{W} g + cg + df = 0.$$

with

$$W = \frac{\Delta(\gamma_1^{1/2} + \gamma_2^{1/2})}{\gamma_1^{1/2} + \gamma_2^{1/2}}, \qquad a = \mathbf{1}_\Omega \omega^2 \gamma_1^{1/2} \gamma_2^{1/2} (\mu_1 + \mu_2), \qquad b = \mathbf{1}_\Omega \omega^2 \gamma_1^{1/2} \gamma_2^{1/2} (\gamma_1 + \gamma_2) \frac{\mu_1^{1/2} + \mu_2^{1/2}}{\gamma_1^{1/2} + \gamma_2^{1/2}},$$

$$\tilde{W} = \frac{\Delta(\mu_1^{1/2} + \mu_2^{1/2})}{\mu_1^{1/2} + \mu_2^{1/2}}, \qquad c = \mathbf{1}_\Omega \omega^2 \mu_1^{1/2} \mu_2^{1/2} (\gamma_1 + \gamma_2), \qquad d = \mathbf{1}_\Omega \omega^2 \mu_1^{1/2} \mu_2^{1/2} (\mu_1 + \mu_2) \frac{\gamma_1^{1/2} + \gamma_2^{1/2}}{\mu_1^{1/2} + \mu_2^{1/2}}.$$

Unique solvability of this elliptic system, and hence uniqueness of the parameters, follows from a suitable unique continuation result.

Uniqueness of the parameters

Unique solvability of this system follows from a unique continuation result.

Lemma

Suppose f and g are compactly supported functions in $H^1(\mathbb{R}^3)$. Then, if f and g satisfy the system of equations

$$
\begin{aligned}
-\Delta f + Wf + af + bg &= 0,\\
-\Delta g + \tilde{W} g + cg + df &= 0,
\end{aligned}
$$

the two functions vanish identically.

Sketch of proof. Let $\zeta \in \mathbb{C}^3$ satisfy $\zeta \cdot \zeta = 0$, and set $u(x) = e^{-\zeta\cdot x} f(x)$ and $v(x) = e^{-\zeta \cdot x} g(x)$. Then u and v are compactly supported functions in $X_\zeta^{1/2}$ and satisfy

$$
\begin{aligned}
-\Delta u + 2\zeta \cdot \nabla u + Wu + au + bv &= 0,\\
-\Delta v + 2\zeta \cdot \nabla v + \tilde{W} v + cv + du &= 0.
\end{aligned}
$$

A sligth modification of the *a priori* estimate can be used to show that this system is uniquely solvable in $X_\zeta^{1/2}$. □

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Thank you!